

Existence and Uniqueness of Global Weak solutions of the Camassa-Holm Equation with a Forcing

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Abstract

In this paper, we study the global well-posedness for the Camassa-Holm(C-H) equation with a forcing in $H^1(\mathbb{R})$ by the characteristic method. Due to the forcing, many important properties to study the well-posedness of weak solutions do not inherit from the C-H equation without a forcing, such as conservation laws, integrability. By exploiting the balance law and some new estimates, we prove the existence and uniqueness of global weak solutions for the C-H equation with a forcing in $H^1(\mathbb{R})$.

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1 Introduction

In this paper, we consider the C-H equation with a forcing in the following form:

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} + ku, \quad (1.1)$$

where $u := u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. The differential operators are defined by $u_t := \frac{\partial u}{\partial t}$, $u_x := \frac{\partial u}{\partial x}$, $u_{xx} := \frac{\partial^2 u}{\partial x^2}$, $u_{txx} := \frac{\partial^3 u}{\partial t \partial x^2}$ and $u_{xxx} := \frac{\partial^3 u}{\partial x^3}$. ku is the forcing term, where $k \in \mathbb{R}$ is a constant. In particular, when $k = 0$, Eq.(1.1) is a well-known integrable equation describing the velocity dynamics of shallow water waves, named Camassa-Holm equation, which models the propagation of unidirectional shallow water waves over a flat bottom by approximating

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the Green-Naghdi equations (see [14]), with formal Hamiltonian derivations provided in [10, 20].

In the last two decades, the C-H equation has attracted many attentions, and a lot of interesting properties have been found, including integrability([10, 12, 15]), existence of peaked solitons and multi-peakons([1, 3]), breaking waves([11]). It should point out that tsunami modelling has been connected to the dynamics of the Camassa-Holm equation(see the discussion in [13, 23]). Among these important properties, the following three conservation laws are crucial in studying the C-H equation.

$$F_1(u) = \int_{\mathbb{R}} (u - u_{xx}) dx, \quad F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2) dx.$$

This implies that the C-H equation can be written in bi-Hamiltonian form which means a second, compatible structure:

$$\partial_t F'_2(u) = -\partial_x F'_3(u),$$

where $'$ denotes the Fréchet derivative. Then, the orbital stability of the peakon (a special global solution of the C-H equation in the form $u(t, x) = ce^{-|x-ct|}$, where $c \in \mathbb{R}$ is wave speed) is well understood in [16, 17, 18]. With regards to the global existence of weak solutions for the Cauchy problem, in [5, 24, 25], the global existence and uniqueness of dispersive solutions is obtained as weak limits of viscous regularizations of the C-H equation in $H^1(\mathbb{R})$. The global existence of conservative solutions is studied in [4, 21] by the new characteristic method. Recently, the uniqueness of conservative solutions is proved in [6] by using the technique of a generalized characteristic method.

However, for Eq.(1.1), due to the existence of the forcing term ku , the structure of Eq.(1.1) changes dramatically. For example, the peakon $u(t, x) = ce^{-|x-ct|}$ is no longer a solution of Eq.(1.1), and in fact, from [22], Eq.(1.1) is not integrable. Most important of all, Eq.(1.1) does not have a Hamiltonian structure. A nature question is arising how to study the well-posedness of weak solutions of Eq.(1.1)? Whether there exists a global weak solution? And how about the uniqueness of the weak solution?

Motivated by those questions, we try to apply the new characteristic method established in [4, 6, 7], to study the global well-posedness for the forcing C-H equation (1.1). We remark that in [4], the authors provided a new characteristic method to study the existence of global conservative solutions for the C-H equation without forcing, relying heavily on the conservation of energy. But this is not essential. The essential structure is the study of a balance law for the C-H equation.

Now, we briefly introduce our results and the key ideas of the proof. We first study the structure of Eq.(1.1). Let $p(x) = \frac{1}{2}e^{-|x|}$, where $x \in \mathbb{R}$. We collect some important properties of $p(x)$ in the following

$$(i) \quad \mathcal{F}[p] = \frac{1}{1+|\xi|^2}.$$

$$(ii) \quad \mathcal{F}[p - p_{xx}] = 1 \text{ implies } p - p_{xx} = \delta(x), \text{ where } \delta \text{ is the Dirac function.}$$

$$(iii) \quad (1 - \partial_x^2)^{-1}u = p * u(x) := \int_{-\infty}^{+\infty} p(x-y)u(y)dy.$$

$$(iv) \quad p * (u - u_{xx}) = u,$$

where \mathcal{F} denotes the Fourier transformation in \mathbb{R} . Then, Eq.(1.1) can be expressed in the following form.

$$u_t + uu_x + P_x - kQ = 0, \quad (1.2)$$

where the singular integral operators P and Q are defined by

$$P := \int_{\mathbb{R}} p(x-y)[u^2 + \frac{1}{2}u_x^2](y)dy \quad \text{and} \quad Q := \int_{\mathbb{R}} p(x-y)u(y)dy. \quad (1.3)$$

For the smooth solutions of Eq.(1.2), we differentiate (1.2) with respect to x .

$$u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 - u^2 + P - kQ_x = 0. \quad (1.4)$$

By multiplying $2u_x$, fortunately, we find that Eq.(1.2) satisfies the following balance law in the following form.

$$(u_x^2)_t + (uu_x^2)_x + 2(-u^2 + P - kQ_x)u_x = 0. \quad (1.5)$$

Indeed, (1.5) is called the balance law due to the third term in (1.5) only includes first-order derivative of u , although some nonlocal operators are involved. In the present paper, we extend the arguments for the C-H equation without a forcing in [4, 6] to the forcing C-H equation (1.2) by using only the balance law (1.5). Actually, there are two main difficulties to study the existence and uniqueness of global weak solutions for the forcing C-H equation (1.2): one is the loss of conservation of energy, that is,

$$\int_{\mathbb{R}} [u^2(t, x) + u_x^2(t, x)]dx \neq \text{Constant}; \quad (1.6)$$

the other is the appearance of a new nonlocal singular operator Q .

We supplement Eq.(1.2) with the initial data

$$u(0, x) = u_0(x). \quad (1.7)$$

Now, we state our main theorem for the existence and uniqueness of global weak solutions for the Cauchy problem (1.2)-(1.7).

Theorem 1.1. *Let $k \in \mathbb{R}$ be a constant. Suppose $u_0 \in H^1(\mathbb{R})$ is an absolutely continuous function on x . Then the Cauchy problem (1.2)-(1.7) admits a global weak solution $u(t, x) \in H^1(\mathbb{R})$ with*

$$\int_{\Gamma} \left\{ -u_x \phi_t - uu_x \phi_x + \left(-\frac{1}{2}u_x^2 - u^2 + P - kQ_x \right) \phi \right\} dx dt + \int_{\mathbb{R}} (u_0)_x \phi(0, x) dx = 0 \quad (1.8)$$

for every test function $\phi \in C_c^1(\Gamma)$ with $\Gamma = \{(t, x) \mid t \in \mathbb{R}, x \in \mathbb{R}\}$. Furthermore, the weak solution satisfies the following properties.

- (i) $u(t, x)$ is $\frac{1}{2}$ -Hölder continuous with respect to t and x , for t in any bounded interval.
- (ii) For every fixed $t \in \mathbb{R}$, the map $t \mapsto u(t, \cdot)$ is Lipschitz continuous under L^2 -norm.
- (iii) The balance law (1.5) is satisfied in the following sense: there exists a family of Radon measures $\{\mu_{(t)}, t \in \mathbb{R}\}$, depending continuously on time and w.r.t the topology of weak convergence of measures, and for every $t \in \mathbb{R}$, the absolutely continuous part of $\mu_{(t)}$ w.r.t. Lebesgue measure has density $u_x^2(t, \cdot)$, which provides a measure-valued solution to the balance law

$$\int_{\Gamma} \{ u_x^2 \phi_t + uu_x^2 \phi_x + 2u_x(u^2 - P + kQ_x) \phi \} dx dt - \int_{\mathbb{R}} (u_0)_x^2 \phi(0, x) dx = 0, \quad (1.9)$$

for every test function $\phi \in C_c^1(\Gamma)$.

- (iv) Some continuous dependence result holds. Consider a sequence of initial data $u_{0,n}$ such that $\|u_{0,n} - u_0\|_{H^1} \rightarrow 0$, as $n \rightarrow +\infty$. Then the corresponding solutions $u_n(t, x)$ converge to $u(t, x)$ uniformly for (t, x) in any bounded sets.

Finally, we prove the uniqueness of the global weak solutions of Eq.(1.2) satisfying the balance law (1.9) in the last section of this paper, as follows.

Theorem 1.2. *Let $k \in \mathbb{R}$ be a constant. For any initial data $u_0 \in H^1(\mathbb{R})$, the Cauchy problem (1.2)-(1.7) has a unique global weak solution satisfying (1.9).*

We point out that in this paper we verified that the methods for both existence and uniqueness of weak solutions used in [4, 6] can be extended to Eq.(1.2) by exploring only a balance law (1.5) instead of a conservation of energy. However, the jump for using a balance law instead of a conservation law is very nontrivial. This idea has also been used for a generalized wave equation with a higher-order nonlinearity in our forthcoming paper [9].

2 The Transferred System

In this paper, we firstly study the global weak solutions of (1.2)-(1.7) in $H^1(\mathbb{R})$, in terms of Bressan and Constantin's arguments in [4](see also[8]). First, we shall use the characteristic method to transfer the quasi-linear Eq.(1.2) to a semi-linear system in terms of smoothing solutions. Then, we prove the existence of global weak solutions for the transferred system by the ODE's argument. Finally, by the inverse transformation, we return to the original problem. More precisely, we introduce a new coordinate (T, Y) defined

$$(t, x) \longrightarrow (T, Y), \quad \text{where} \quad T = t, \quad Y = \int_0^{x_0(Y)} (1 + (u_0)_x^2) dx. \quad (2.1)$$

The equation of the characteristic is

$$\frac{dx(t, Y)}{dt} = u(t, x(t, Y)) \quad \text{with} \quad x(0, Y) = x_0(Y). \quad (2.2)$$

Here, $Y = Y(t, x)$ is a characteristic coordinate and satisfies $Y_t + uY_x = 0$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$. After some computations, we see that for any smoothing function $f := f(T, Y(t, x))$ have the following properties, under the new coordinates (T, Y) :

$$f_T = f_T (T_t + uT_x) + f_Y (Y_t + uY_x) = f_t + uf_x,$$

$$f_x = f_T T_x + f_Y Y_x = f_Y Y_x.$$

Take the transformation:

$$v := 2 \arctan u_x \quad \text{and} \quad \xi := \frac{1 + u_x^2}{Y_x}. \quad (2.3)$$

Under this transformation, we see that

$$u_x = \tan \frac{v}{2}, \quad 1 + u_x^2 = \sec^2 \frac{v}{2}, \quad \frac{1}{1 + u_x^2} = \cos^2 \frac{v}{2}, \quad (2.4)$$

$$\frac{u_x^2}{1 + u_x^2} = \sin^2 \frac{v}{2}, \quad \frac{u_x}{1 + u_x^2} = \frac{1}{2} \sin v, \quad x_Y = \frac{\xi}{1 + u_x^2} = \cos^2 \frac{v}{2} \cdot \xi. \quad (2.5)$$

Then, we will consider Eq.(1.2) under the new characteristic coordinate (T, Y) . $u_T = u_t + uu_x = -P_x + kQ$. From (1.4), we deduce that

$$\begin{aligned} v_T &= \frac{2}{1+u_x^2} (u_x)_T \\ &= \frac{2}{1+u_x^2} (u_{xt} + uu_{xx}) \\ &= \frac{2}{1+u_x^2} \left(-\frac{1}{2} u_x^2 + u^2 - P + kQ_x \right) \\ &= -\sin^2 \frac{v}{2} + 2 \cos^2 \frac{v}{2} (u^2 - P + kQ_x). \end{aligned}$$

For ξ_T , from $Y_t + uY_x = 0$, we have $Y_{tx} + uY_{xx} = -u_x Y_x$. Then, (1.5) implies

$$\begin{aligned} \xi_T &= \frac{2u_x}{Y_x} (u_{tx} + uu_{xx}) + \frac{-(1+u_x^2)}{Y_x^2} (Y_{tx} + uY_{xx}) \\ &= \frac{1}{Y_x} (2u_x u_{tx} + 2uu_x u_{xx} + u_x^3 + u_x) \\ &= 2 \frac{1+u_x^2}{Y_x} \frac{u_x}{1+u_x^2} \left[\frac{(u_x^2)_t + (uu_x^2)_x}{2u_x} + \frac{1}{2} \right] \\ &= \xi \sin v \left(\frac{1}{2} + u^2 - P + kQ_x \right). \end{aligned}$$

In conclusion, from the new transformation (2.3), (1.2) becomes

$$\begin{cases} u_T = -P_x + kQ, \\ v_T = -\sin^2 \frac{v}{2} + 2 \cos^2 \frac{v}{2} (u^2 - P + kQ_x), \\ \xi_T = \xi \sin v \left(\frac{1}{2} + u^2 - P + kQ_x \right). \end{cases} \quad (2.6)$$

We supplement (2.6) with the initial data under the new coordinate (T, Y)

$$\begin{cases} u(0, Y) = u_0(x_0(Y)) \\ v(0, Y) = 2 \arctan(u_0)_x(x_0(Y)), \\ \xi(0, Y) = 1. \end{cases} \quad (2.7)$$

Next, we show the expression of P , P_x , Q and Q_x under the new coordinate (T, Y) . It follows from the last formula in (2.5) that

$$x(T, Y) - x(T, Y') = \int_{Y'}^Y \cos^2 \frac{v(T, s)}{2} \cdot \xi(t, s) ds.$$

We take $x = x(T, Y')$, then $dx = \frac{\xi}{1+u_x^2} dY'$, the validity of which will be checked below. Thus, P , P_x , Q and Q_x has the following form under (T, Y) .

$$\begin{aligned} P(T, Y) &= P(T, x(T, Y)) = p * [u^2 + \frac{1}{2}u_x^2] = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x(T, Y)-x|} [u^2 + \frac{1}{2}u_x^2] dx \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} [u^2 \cos^2 \frac{v(Y')}{2} + \frac{1}{2} \sin^2 \frac{v(Y')}{2}] \xi(Y') dY', \end{aligned} \quad (2.8)$$

$$\begin{aligned} P_x(T, Y) &= P_x(T, x(T, Y)) = p_x * [\frac{2\lambda+1}{2} u^\lambda u_x^2 + u^{\lambda+2}] \\ &= \frac{1}{2} (\int_Y^{+\infty} - \int_{-\infty}^Y) e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} \\ &\quad \cdot [u^2 \cos^2 \frac{v(Y')}{2} + \frac{1}{2} \sin^2 \frac{v(Y')}{2}] \xi(Y') dY', \end{aligned} \quad (2.9)$$

$$Q(T, Y) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} u \cos^2 \frac{v(Y')}{2} \xi(Y') dY', \quad (2.10)$$

$$Q_x(T, Y) = \frac{1}{2} (\int_Y^{+\infty} - \int_{-\infty}^Y) e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} u \cos^2 \frac{v(Y')}{2} \xi(Y') dY'. \quad (2.11)$$

We remark that the equivalent semi-linear system (2.6) is invariant under translation by 2π in v . It would be more precise to use e^{iv} as variable. For simplicity, we use $v \in [-\pi, \pi]$ with endpoints identified.

3 Local Existence of the Transferred System

In this section, we will prove the existence of weak solutions of the transferred system (2.6)-(2.7) by the contracting mapping theory. Here, the work space X is defined by $X := H^1(\mathbb{R}) \times [L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})] \times L^\infty(\mathbb{R})$ with its norm $\|(u, v, \xi)\|_X = \|u\|_{H^1} + \|v\|_{L^2} + \|v\|_{L^\infty} + \|\xi\|_{L^\infty}$. As usual, we shall prove the existence of a fixed point of the integral transformation: $\Phi(u, v, \xi) = (\tilde{u}, \tilde{v}, \tilde{\xi})$, where $(\tilde{u}, \tilde{v}, \tilde{\xi})$ is defined by

$$\begin{cases} \tilde{u}(T, Y) = u_0(x_0(Y)) + \int_0^T (-P_x(\tau, Y) + kQ(\tau, Y)) d\tau, \\ \tilde{v}(T, Y) = 2 \arctan(u_0)_x(x_0(Y)) + \int_0^T [-\sin^2 \frac{v}{2} + 2 \cos^2 \frac{v}{2} (u^2 - P + kQ_x)] d\tau, \\ \tilde{\xi}(T, Y) = 1 + \int_0^T [\xi \sin v (\frac{1}{2} + u^2 - P + kQ_x)] d\tau, \end{cases}$$

where P , P_x , Q and Q_x are defined by (2.8)-(2.11).

As the standard ODE's theory in the Banach space, we shall prove that all functions on the right hand side of (2.6) are locally Lipschitz continuous with respect to (u, v, ξ) in X . Then, we can obtain the local existence of weak solutions of (2.6)-(2.7), as follows.

Proposition 3.1. *Let $k \in \mathbb{R}$ be a constant. If $u_0 \in H^1(\mathbb{R})$, then the Cauchy problem (2.6)-(2.7) has a unique solution on the interval $[0, T]$.*

Proof. Let $K \subset X$ be a bounded domain and defined by

$$K = \{(u, v, \xi) \mid \|u\|_{H^1} \leq A, \|v\|_{L^2} \leq B, \|v\|_{L^\infty} \leq \frac{3\pi}{2}, \xi(Y) \in [C^-, C^+]\}$$

for a.e. $Y \in \mathbb{R}$ and constants $A, B, C^-, C^+ > 0$. Then, if the mapping $\Phi(u, v, \xi)$ is Lipschitz continuous on K , then $\Phi(u, v, \xi)$ has a fixed point by contraction argument and the existence of local solutions will be followed.

In order to do so, first, it follows from the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with any $q > 1$ that $\|u\|_{L^\infty} \leq C\|u\|_{H^1}$. Due to the uniform boundedness of v and ξ , the following maps:

$$\sin^2 \frac{v}{2}, u^2 \cos^2 \frac{v}{2}, \xi \sin v, u^2 \xi \sin v$$

are all Lipschitz continuous as maps from K into $L^2(\mathbb{R})$, as well as from K into $L^\infty(\mathbb{R})$. In order to estimate the singular integrals in P, P_x, Q and Q_x , we observe that $e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} \leq 1$. But this is not enough to control the above singular integrals. Here, we claim the following estimate: If $\|v\|_{L^2} \leq B$, then for any $(u, v, \xi) \in K, f \in L^q(\mathbb{R})$ with $q \geq 1$

$$\left\| \int_{-\infty}^{+\infty} e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} f(Y') dY' \right\|_{L^q} \leq \|g\|_{L^1} \|f\|_{L^q}, \quad (3.1)$$

where $g(z) = \min\{1, e^{\frac{C^-}{2}(\frac{B^2}{2}-|z|)}\}$ and $\|g\|_{L^1} = B^2 + \frac{4}{C^-}$. Indeed, when $\frac{\pi}{4} \leq |\frac{v(Y)}{2}| \leq \frac{3\pi}{4}$, we get $\sin^2 \frac{v(Y)}{2} \geq \frac{1}{2}$. Then, for any $(u, v, \xi) \in K$,

$$\text{meas}\{Y \in \mathbb{R} \mid |\frac{v(Y)}{2}| \geq \frac{\pi}{4}\} \leq 2 \int_{\{Y \in \mathbb{R} \mid \sin^2 \frac{v(Y)}{2} \geq \frac{1}{2}\}} \sin^2 \frac{v(Y)}{2} dY \leq \frac{1}{2} \|v\|_{L^2}^2.$$

For any $z_1 < z_2$, we deduce that

$$\begin{aligned} \int_{z_1}^{z_2} \cos^2 \frac{v(z)}{2} \cdot \xi(z) dz &\geq \frac{C^-}{2} (|z_2 - z_1| - \int_{\{z \in [z_1, z_2] \mid |\frac{v(Y)}{2}| \geq \frac{\pi}{4}\}} dz) \\ &\geq \frac{C^-}{2} (|z_2 - z_1| - \frac{1}{2} B^2). \end{aligned}$$

The above estimate guarantees proper control on the singular integrals in P, P_x, Q and Q_x , which decreases quickly as $|Y - Y'| \rightarrow +\infty$. Therefore, taking $g(z) = \min\{1, e^{\frac{C^-}{2}(\frac{B^2}{2}-|z|)}\}$, for every $q \geq 1$, we see that

$$\left| \int_{-\infty}^{+\infty} e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} f(Y') dY' \right| \leq |g * f(Y)|.$$

(3.1) follows from the Young inequality. In addition,

$$\|g\|_{L^1} = \int_{-\frac{B^2}{2}}^{\frac{B^2}{2}} 1dz + \int_{\frac{B^2}{2}}^{+\infty} e^{\frac{C^-}{2}(\frac{B^2}{2}-z)} dz + \int_{-\infty}^{-\frac{B^2}{2}} e^{\frac{C^-}{2}(\frac{B^2}{2}+z)} dz = B^2 + \frac{4}{C^-}.$$

This completes the proof of Claim (3.1).

Thirdly, by (3.1), we give a priori bounds on P , P_x , Q and Q_x , which implies P , P_x , Q and $Q_x \in H^1(\mathbb{R})$. More precisely, we deduce that P , $\partial_Y P$, P_x , $\partial_Y P_x$, Q , $\partial_Y Q$, Q_x , $\partial_Y Q_x \in L^2(\mathbb{R})$. Indeed, we give estimates for Q , and the estimates for P can be obtain by similar argument. By applying (3.1), we deduce that

$$\partial_Y Q = \frac{1}{2} \left(\int_Y^{+\infty} - \int_{-\infty}^Y \right) e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} \text{sign}(Y' - Y) \cdot [u \cos^4 \frac{v(Y')}{2}] \cdot \xi^2(Y') dY', \quad (3.2)$$

$$\partial_Y Q_x = -u\xi \cos^2 \frac{v(Y')}{2} + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} \cdot [u \cos^4 \frac{v(Y')}{2}] \cdot \xi^2(Y') dY'. \quad (3.3)$$

Then, we have

$$\|Q\|_{L^2} \leq \frac{C^+}{2} \|g\|_{L^1} \|u\|_{L^2}, \quad \|\partial_Y Q\|_{L^2} \leq (C^+)^2 \|g\|_{L^1} \|u\|_{L^2}, \quad (3.4)$$

$$\|Q_x\|_{L^2} \leq C^+ \|g\|_{L^1} \|u\|_{L^2}, \quad \|\partial_Y Q_x\|_{L^2} \leq (C^+ + \frac{(C^+)^2}{2} \|g\|_{L^1}) \|u\|_{L^2}. \quad (3.5)$$

By the same arguments, we can obtain the similar estimates for P in the following.

$$\begin{aligned} \|P\|_{L^2} &\leq \frac{C^+}{2} \|g\|_{L^1} (\|u\|_{L^\infty} \|u\|_{L^2} + \frac{1}{8} \|v\|_{L^2}), \\ \|\partial_Y P\|_{L^2} &\leq (C^+)^2 \|g\|_{L^1} (\|u\|_{L^\infty} \|u\|_{L^2} + \frac{1}{8} \|v\|_{L^2}), \\ \|P_x\|_{L^2} &\leq C^+ \|g\|_{L^1} (\|u\|_{L^\infty} \|u\|_{L^2} + \frac{1}{8} \|v\|_{L^2}), \\ \|\partial_Y P_x\|_{L^2} &\leq (C^+ + (C^+)^2 \|g\|_{L^1}) (\|u\|_{L^\infty} \|u\|_{L^2} + \frac{1}{8} \|v\|_{L^2}). \end{aligned} \quad (3.6)$$

Using the Sobolev inequality, we have $\|u\|_{L^\infty} \leq C\|u\|_{H^1}$ and this implies P , P_x , Q and $Q_x \in H^1(\mathbb{R})$. Then, according to the above discussion, we show that the mapping $\Phi(u, v, \xi)$ is from K to K .

Fourthly, we shall prove the mapping $\Phi(u, v, \xi)$ is Lipschitz continuity with respect to (u, v, ξ) . More precisely, we need to prove that the following partial derivatives $\frac{\partial P}{\partial u}$, $\frac{\partial P}{\partial v}$, $\frac{\partial P}{\partial \xi}$, $\frac{\partial P_x}{\partial u}$, $\frac{\partial P_x}{\partial v}$, $\frac{\partial P_x}{\partial \xi}$, $\frac{\partial Q}{\partial u}$, $\frac{\partial Q}{\partial v}$, $\frac{\partial Q}{\partial \xi}$, $\frac{\partial Q_x}{\partial u}$, $\frac{\partial Q_x}{\partial v}$, $\frac{\partial Q_x}{\partial \xi}$ are uniformly bounded as (u, v, ξ) range inside the domain K . We observe these derivatives are bounded linear operators from the appropriate spaces into $H^1(\mathbb{R})$. For sake of illustration, we shall give the detail estimates for Q .

And the estimates for P will be similar obtained. In other words, we shall prove $\frac{\partial Q}{\partial u}$ and $\frac{\partial Q_x}{\partial u}$ are linear operators from $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$; $\frac{\partial Q}{\partial v}$ and $\frac{\partial Q_x}{\partial v}$ are linear operators from $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ into $L^2(\mathbb{R})$; $\frac{\partial Q}{\partial \xi}$ and $\frac{\partial Q_x}{\partial \xi}$ are linear operators from $L^\infty(\mathbb{R})$ into $L^2(\mathbb{R})$.

Here, we shall take $\frac{\partial Q}{\partial u}$ and $\frac{\partial(\partial_Y Q)}{\partial u}$ as examples to illustrate the main idea. Indeed, $\forall \phi \in H^1(\mathbb{R})$,

$$\begin{aligned} \left\| \frac{\partial Q}{\partial u} \cdot \phi \right\|_{L^2} &\leq \frac{1}{2} \|\xi\|_{L^\infty} \|g * \phi\|_{L^2} \leq \frac{C^+}{2} \|g\|_{L^1} \|\phi\|_{H^1}, \\ \left\| \frac{\partial(\partial_Y Q)}{\partial u} \cdot \phi \right\|_{L^2} &\leq \|\xi\|_{L^\infty}^2 \|g * \phi\|_{L^2} \leq (C^+)^2 \|g\|_{L^1} \|\phi\|_{H^1}, \\ \left\| \frac{\partial Q_x}{\partial u} \cdot \phi \right\|_{L^2} &\leq \|\xi\|_{L^\infty} \|g * \phi\|_{L^2} \leq C^+ \|g\|_{L^1} \|\phi\|_{H^1}, \\ \left\| \frac{\partial(\partial_Y Q_x)}{\partial u} \cdot \phi \right\|_{L^2} &\leq \|\xi\|_{L^\infty} \|\phi\|_{L^2} + \frac{1}{2} \|\xi\|_{L^\infty}^2 \|g * \phi\|_{L^2} \leq (C^+ + \frac{(C^+)^2}{2} \|g\|_{L^1}) \|\phi\|_{H^1}. \end{aligned}$$

Thus, the linear operator $\frac{\partial Q}{\partial u}$, $\frac{\partial(\partial_Y Q)}{\partial u}$, $\frac{\partial Q_x}{\partial u}$ and $\frac{\partial(\partial_Y Q_x)}{\partial u}$ are bounded from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$.

Finally, we prove the mapping $\Phi(u, v, \xi)$ is uniformly Lipschitz continuity on a neighborhood K of the initial data in the space X . Then, we can apply the standard theory of ODE's local existence in Banach spaces, there exist a solutions to the Cauchy problem (2.6)-(2.7) on small time interval $[-T, T]$. This completes the proof of Proposition 3.1. \square

4 Global Existence of the Transferred System

In this section, we shall prove that the local solution for the Cauchy problem (2.6)-(2.7) can be extended to the global one. To ensure this, we need to show that there exists a bound $C(E_0) > 0$ such that

$$\|u(T)\|_{H^1} + \|v(T)\|_{L^2} + \|v(T)\|_{L^\infty} + \|\xi(T)\|_{L^\infty} + \left\| \frac{1}{\xi(T)} \right\|_{L^\infty} \leq C(E_0), \quad (4.1)$$

for all $T \in \mathbb{R}$. Then, we obtain the following proposition.

Proposition 4.1. *Let $k \in \mathbb{R}$ be a constant. If the initial data $u_0 \in H^1(\mathbb{R})$, then the Cauchy problem (2.6)-(2.7) has a unique solution, defined for all time $T \in \mathbb{R}$.*

Proof. In Proposition 3.1, we have proved the existence of the local solution for the Cauchy problem (2.6)-(2.7). First of all, when $t = 0$, from the transformation (2.3)-(2.5), we see that $Y_x(0) = 1 + (u_0)_x^2$, then

$$u_Y(0, Y) = \frac{(u_0)_x}{1 + (u_0)_x^2} = \frac{1}{2} \sin v(0, Y) \quad \text{and} \quad \xi(0, Y) = 1.$$

We claim that for all time T ,

$$u_Y = \frac{1}{2}\xi \sin v = \xi \sin \frac{v}{2} \cos \frac{v}{2}. \quad (4.2)$$

Indeed, from the first identity in (2.6), we have

$$\begin{aligned} (u_Y)_T = (u_T)_Y &= -(P_x)_Y + kQ_Y = \frac{-P_{xx} + kQ_x}{Y_x} \\ &= \xi[u^2 \cos^2 \frac{v}{2} + \frac{1}{2} \sin^2 \frac{v}{2} + \cos^2 \frac{v}{2}(-P + kQ_x)]. \end{aligned}$$

From the last two identity in (2.6), we see that for all time T

$$\begin{aligned} (\frac{1}{2}\xi \sin v)_T &= \frac{1}{2} \sin v \xi_T + \frac{1}{2}\xi \cos v v_T \\ &= 2u^2 \xi \sin^2 \frac{v}{2} \cos^2 \frac{v}{2} + 2\xi \sin^2 \frac{v}{2} \cos^2 \frac{v}{2}(-P + kQ_x) \\ &\quad + \frac{1}{2}\xi \sin^2 \frac{v}{2} + 2u^2 \xi \cos^4 \frac{v}{2} + 2\xi \cos^4 \frac{v}{2}(-P + kQ_x) \\ &\quad - \xi u^2 \cos^2 \frac{v}{2} - \xi \cos^2 \frac{v}{2}(-P + kQ_x) \\ &= (u_Y)_T. \end{aligned}$$

Then, it follows from $\xi(0, Y) = 1$ that (4.2) is true for all time T . Next, we estimate the functional $E(T) := \int_{\mathbb{R}} (u^2 \xi \cos^2 \frac{v}{2} + \xi \sin^2 \frac{v}{2}) dY$ under the new coordinate. From (2.6), we have

$$\begin{aligned} &\frac{d}{dT} \int_{\mathbb{R}} (u^2 \xi \cos^2 \frac{v}{2} + \xi \sin^2 \frac{v}{2}) dY \\ &= \int_{\mathbb{R}} \{ (u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2}) \xi_T + 2u \xi \cos^2 \frac{v}{2} u_T + \xi \sin \frac{v}{2} \cos \frac{v}{2} (1 - u^2) v_T \} dY \\ &= \int_{\mathbb{R}} \{ u^2 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} + 2u^4 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} + 2u^2 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} (-P + kQ_x) \\ &\quad + \xi \sin^3 \frac{v}{2} \cos \frac{v}{2} + 2u^2 \xi \sin^3 \frac{v}{2} \cos \frac{v}{2} + 2\xi \sin^3 \frac{v}{2} \cos \frac{v}{2} (-P + kQ_x) \\ &\quad + 2u \xi \cos^2 \frac{v}{2} (-P_x + kQ) - \xi \sin^3 \frac{v}{2} \cos \frac{v}{2} + 2u^2 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} \\ &\quad + 2\xi \sin \frac{v}{2} \cos^3 \frac{v}{2} (-P + kQ_x) + u^2 \xi \sin^3 \frac{v}{2} \cos \frac{v}{2} - 2u^4 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} \\ &\quad - 2u^2 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} (-P + kQ_x) \} dY \\ &= \int_{\mathbb{R}} \{ 3u^2 \xi \sin \frac{v}{2} \cos \frac{v}{2} + 2u \xi \cos^2 \frac{v}{2} (-P_x + kQ) + 2\xi \sin \frac{v}{2} \cos \frac{v}{2} (-P + kQ_x) \} dY \\ &= \int_{\mathbb{R}} 3u^2 u_Y dY + 2 \int_{\mathbb{R}} [u(-P + kQ_x)]_Y dY + 2k \int_{\mathbb{R}} u^2 \xi \cos^2 \frac{v}{2} dY \\ &= 2k \int_{\mathbb{R}} u^2 \xi \cos^2 \frac{v}{2} dY, \end{aligned} \quad (4.3)$$

where in the last two estimates, we use the facts: for any $u \in H^1(\mathbb{R})$,

$$\lim_{|Y| \rightarrow +\infty} u(Y) = 0, \quad (u^3)_Y = 3u^2 u_Y = 3u^2 \xi \sin \frac{v}{2} \cos \frac{v}{2} \text{ and}$$

$$2[u(-P + kQ_x)]_Y = 2\xi \sin \frac{v}{2} \cos \frac{v}{2} (-P + kQ_x) + 2u \xi \cos^2 \frac{v}{2} (-P_x + kQ) + 2ku^2 \xi \cos^2 \frac{v}{2}.$$

Then, from (4.3), we can deduce that there exists a constant $D_0 := D_0(E_0, T) > 0$ such that the new energy $E(T)$ has a priori on bounded intervals of time by the Gronwall inequality. That is, for any T on bounded intervals,

$$E(T) \leq D_0^2. \quad (4.4)$$

Secondly, we estimate $\|u(T)\|_{L^\infty}$, $\|\xi(T)\|_{L^\infty}$, $\|\frac{1}{\xi(T)}\|_{L^\infty}$ and $\|v(T)\|_{L^\infty}$ with respect to T . For all the solutions of the Cauchy problem (2.6)-(2.7), it follows from (4.2) and (4.4) that

$$\begin{aligned} \sup_{Y \in \mathbb{R}} |u^2(T, Y)| &\leq \int_{\mathbb{R}} |(u^2)_Y| dY = \int_{\mathbb{R}} |u \sin \frac{v}{2} \cos \frac{v}{2}| \xi dY \\ &\leq \int_{\mathbb{R}} (u^2 \xi \cos^2 \frac{v}{2} + \xi \sin^2 \frac{v}{2}) dY, \end{aligned}$$

which implies that $\|u(T)\|_{L^\infty}$ has a priori bound D_0 . More precisely, we have the following estimate:

$$\|u(T)\|_{L^\infty} \leq D_0. \quad (4.5)$$

By the definition of P , P_x , Q and Q_x in (2.8)-(2.11), using (4.4) and (4.5), we deduce that

$$\|P(T)\|_{L^\infty}, \|P_x(T)\|_{L^\infty} \leq \|u^2 \xi \cos^2 \frac{v}{2} + \frac{1}{2} \xi \sin^2 \frac{v}{2}\|_{L^1} \leq D_0^2, \quad (4.6)$$

$$\|Q(T)\|_{L^\infty}, \|Q_x(T)\|_{L^\infty} \leq \|e^{-|\int_Y^{Y'} \cos^2 \frac{v}{2} \cdot \xi ds|} \xi(T)\|_{L^1} \|u\|_{L^\infty} \leq CD_0. \quad (4.7)$$

Inject (4.6) and (4.7) into the last identity in (2.6), we see that

$$|\xi_T| \leq \left\| \frac{1}{2} + u^2 - P + kQ_x \right\|_{L^\infty} \xi \leq \left(\frac{1}{2} + 2D_0^2 + kCD_0 \right) \xi.$$

And from the initial condition $\xi(0, Y) = 1$, we get

$$\frac{1}{D_1} \leq \xi(T) \leq D_1, \quad (4.8)$$

where $D_1 = e^{(\frac{1}{2} + 2D_0^2 + kCD_0)|T|}$. From the second identity in (2.6), we have

$$\frac{d}{dT} \|v(T)\|_{L^\infty} \leq 2(\|u(T)\|_{L^\infty}^2 + \|P\|_{L^\infty} + k\|Q_x\|_{L^\infty}) \leq 2D_0(kC + 2D_0).$$

Denote $D_2 = 2D_0(kC + 2D_0)$. Then, we have

$$\|v(T)\|_{L^\infty} \leq e^{D_2 T}. \quad (4.9)$$

Thirdly, we estimate $\|u(T)\|_{H^1}$ with respect to T . From $u_T = -P_x + kQ$,

$$\frac{d}{dT}\|u(T)\|_{L^2}^2 = 2 \int_{\mathbb{R}} uu_T dY \leq 2\|u(T)\|_{L^\infty}(\|P_x\|_{L^1} + k\|Q\|_{L^1}), \quad (4.10)$$

$$\frac{d}{dT}\|(u(T))_Y\|_{L^2}^2 = 2 \int_{\mathbb{R}} u_Y(u_T)_Y dY \leq 2\|u_Y\|_{L^\infty}(\|(P_x)_Y\|_{L^1} + k\|Q_Y\|_{L^1}). \quad (4.11)$$

In order to obtain the boundness of $\|u(T)\|_{H^1}$, we need the following lemma to estimate $\|P_x\|_{L^1}$, $\|Q\|_{L^1}$, $\|(P_x)_Y\|_{L^1}$ and $\|Q_Y\|_{L^1}$.

Lemma 4.2. *Let $f \in L^q$ with $q \geq 1$. If $\|\xi \sin^2 \frac{v(T)}{2}\|_{L^1} \leq D_0^2$, then*

$$\left\| \int_{-\infty}^{+\infty} e^{-|\int_Y^{Y'} \cos^2 \frac{v(s)}{2} \cdot \xi(s) ds|} f(Y') dY' \right\|_{L^q} \leq \|h * f\|_{L^q} \leq \|h\|_{L^1} \|f\|_{L^q}, \quad (4.12)$$

where $h(z) = \min\{1, e^{\frac{1}{2D_1}(2D_1D_0^2 - |z|)}\}$ and $\|h\|_{L^1} = 4D_1D_0^2 + 4D_1$.

Proof. We remark that $\|\xi \sin^2 \frac{v(T)}{2}\|_{L^1} \leq D_0^2$ does not imply that $\|v(T)\|_{L^2}$ is bounded, which is the main difficulty. As proved in above, we see that $\|v(T)\|_{L^\infty}$ is bounded, and we may assume $|v| \leq \frac{3\pi}{2}$ because $v(T)$ is invariant under multiplier of 2π . Then, when $\frac{\pi}{4} \leq |\frac{v(Y)}{2}| \leq \frac{3\pi}{4}$, we have $\sin^2 \frac{v(Y)}{2} \geq \frac{1}{2}$. And, we deduce that

$$\begin{aligned} & \text{meas}\{Y \in \mathbb{R} \mid |\frac{v(Y)}{2}| \geq \frac{\pi}{4}\} \\ & \leq \text{meas}\{Y \in \mathbb{R} \mid \sin^2 \frac{v(Y)}{2} \geq \frac{1}{2}\} \\ & \leq 2 \int_{\{Y \in \mathbb{R} \mid \sin^2 \frac{v(Y)}{2} \geq \frac{1}{2}\}} D_1 \xi \sin^2 \frac{v(Y)}{2} dY \\ & \leq 2D_1D_0^2. \end{aligned} \quad (4.13)$$

For any $z_1 < z_2$, we have

$$\begin{aligned} \int_{z_1}^{z_2} \cos^2 \frac{v(z)}{2} \cdot \xi(z) dz & \geq \frac{1}{2D_1} \left(\int_{z_1}^{z_2} 1 dz - \int_{\{z \in [z_1, z_2] \mid |\frac{v(Y)}{2}| \geq \frac{\pi}{4}\}} 1 dz \right) \\ & \geq \frac{1}{2D_1} (|z_2 - z_1| - 2D_1D_0^2). \end{aligned}$$

Let $h(z) = \min\{1, e^{\frac{1}{2D_1}(2D_1D_0^2 - |z|)}\}$. Then, for every $q \geq 1$, (4.12) follows from the Young inequality. In addition, by some computation, we have

$$\begin{aligned} \|h\|_{L^1} & = \int_{-2D_1D_0^2}^{2D_1D_0^2} 1 dz + \int_{2D_1D_0^2}^{+\infty} e^{\frac{2D_1D_0^2 - z}{2D_1}} dz + \int_{-\infty}^{-2D_1D_0^2} e^{\frac{2D_1D_0^2 + z}{2D_1}} dz \\ & = 4D_1D_0^2 + 4D_1. \end{aligned}$$

□

Now, we return to the proof of Proposition 4.1. By using Lemma 4.2, we have the following estimates:

$$\begin{aligned} \|P_x\|_{L^1} &\leq D_0^2 \|h\|_{L^1}, & \|(P_x)_Y\|_{L^1} &\leq (1 + D_1) D_0^2 \|h\|_{L^1}, \\ \|Q\|_{L^1} &\leq D_1 D_0 \|h\|_{L^1}, & \|Q_Y\|_{L^1} &\leq D_1^2 D_0 \|h\|_{L^1}. \end{aligned} \quad (4.14)$$

Injecting (4.14) into (4.10) and (4.11), we can deduce that $\|u(T)\|_{L^2}^2$ and $\|(u(T))_Y\|_{L^2}^2$ are bounded on any bounded interval of time T . That is, there exists a constant $D_3 := D_3(E_0, T) > 0$ such that

$$\|u(T)\|_{H^1} \leq D_3. \quad (4.15)$$

Finally, we estimate $\|v(T)\|_{L^2}$ with respect to T . Multiplying $v(T)$ to the second identity in (2.6) and integrating, we deduce that

$$\begin{aligned} \frac{d}{dT} \|v(T)\|_{L^2}^2 &= -2 \int_{\mathbb{R}} \sin^2 \frac{v}{2} v dY + 4 \int_{\mathbb{R}} \cos^2 \frac{v}{2} (u^2 - P + kQ_x) v dY \\ &\leq 2D_1 \|v(T)\|_{L^\infty} \|\xi \sin^2 \frac{v(T)}{2}\|_{L^1} \\ &\quad + 4(\|u(T)\|_{L^\infty} \|u(T)\|_{L^2} + \|P\|_{L^2} + k\|Q_x\|_{L^2}) \|v(T)\|_{L^2}. \end{aligned} \quad (4.16)$$

Inject (4.4), (4.5), (4.8), (4.9) and (4.15) into the estimates in $\|P\|_{L^2}$ and $\|Q_x\|_{L^2}$. We have

$$\|P\|_{L^2}^2 \leq D_1 \left(\frac{1}{2} + D_0^2 \right) D_0^2 \|h\|_{L^1}^2, \quad \|Q_x\|_{L^2}^2 \leq D_1^2 D_0^2 \|h\|_{L^1}^2. \quad (4.17)$$

Then, injecting (4.17) into (4.16) and using the Gronwall inequality, we can deduce that $\|v(T)\|_{L^2}^2$ is bounded on any bounded interval of time T . That is, there exists a constant $D_4 := D_4(E_0, T) > 0$ such that

$$\|v(T)\|_{L^2} \leq D_4. \quad (4.18)$$

This completes the proof of Proposition 4.1. \square

Remark 4.3. We defined the set of times

$$N := \{T \geq 0 \mid \text{meas}\{Y \in \mathbb{R}; v(T, Y) = -\pi\} > 0\}.$$

Then, we claim that the measure of N must be 0. That is $\text{meas}(N) = 0$. Indeed, from the second identity in the semi-linear system (2.6), we see that $v_T = -1$ provided $\cos v = -1$. Then, it follows from the absolute continuity of v that we can find $\delta > 0$ such that $v_T \leq M < 0$ wherever $1 + \cos v < \delta$. On the other hand, we remark the fact that $\|v(T)\|_{L^2}$ remains bounded on bounded time intervals, and then we can obtain $\text{meas}(N) = 0$. If not, then we have $\int \int_{\{v(T, Y) = -\pi\}} v_T dY dT < 0$. This is contradict to the fact $v_T = 0$ a.e. on $\{v(T, Y) = -\pi\}$, in terms of the absolute continuity of the map $T \rightarrow v(T, Y)$ at every fixed $Y \in \mathbb{R}$.

5 Existence of the Global Weak Solution

Now, we start with a global solution (u, v, ξ) to (2.6) obtained in Proposition 4.1. We define x and t as functions of T and Y , where $t = T$ and

$$x(T, Y) := x_0(Y) + \int_0^T u(\tau, Y) d\tau. \quad (5.1)$$

For each fixed Y , the function $T \mapsto x(T, Y)$ thus provides a solution to the Cauchy problem

$$\frac{d}{dT} x(T, Y) = u(T, x(T, Y)), \quad x(0, Y) = x_0(Y).$$

Then, by taking

$$u(T, x) := u(T, Y) \quad \text{provided} \quad x(T, Y) = x, \quad (5.2)$$

we can prove that $u(t, x)$ is a solution of Eq.(1.2). Then, we can complete the proof of Theorem 1.1.

Proof. First, we prove that the function $u = u(t, x)$ is well-defined. From (4.5), we see that $|u(T, Y)| \leq D_0$. By (5.1), we get

$$x_0(Y) - D_0 T \leq x(T, Y) \leq x_0(Y) + D_0 T.$$

From $Y = \int_0^{x_0(Y)} (1 + (u_0)_x^2) dx$, we have $\lim_{Y \rightarrow \pm\infty} x_0(Y) = \pm\infty$, this yields the image of the map $(T, Y) \mapsto (T, x(T, Y))$ is the entire plane \mathbb{R}^2 . We claim

$$x_Y = \cos^2 \frac{v}{2} \cdot \xi \quad (5.3)$$

for all t and a.e. $Y \in \mathbb{R}$. Indeed, from (2.6), we deduce that

$$\begin{aligned} & \frac{d}{dT} \cos^2 \frac{v}{2} \cdot \xi \\ &= \cos^2 \frac{v}{2} \cdot \xi_T - \xi \cos \frac{v}{2} \sin \frac{v}{2} \cdot v_T \\ &= \xi \cos^3 \frac{v}{2} \sin \frac{v}{2} + 2u^2 \xi \cos^3 \frac{v}{2} \sin \frac{v}{2} + 2\xi \cos^3 \frac{v}{2} \sin \frac{v}{2} (-P + kQ_x) \\ & \quad + \xi \sin^3 \frac{v}{2} \cos \frac{v}{2} - 2u^2 \xi \sin \frac{v}{2} \cos^3 \frac{v}{2} - 2\xi \sin \frac{v}{2} \cos^3 \frac{v}{2} (-P + kQ_x) \\ &= \xi \sin \frac{v}{2} \cos \frac{v}{2} \\ &= u_Y. \end{aligned}$$

And by differentiating (5.1) with T and Y , we deduce that

$$\frac{d}{dT} x_Y = \frac{d}{dT} ((x_0)_Y + \int_0^T u_Y d\tau) = u_Y = \frac{d}{dT} \cos^2 \frac{v}{2} \cdot \xi.$$

Moreover, from the fact that $x \mapsto 2\arctan(u_0)_x(x)$ is measurable, we see that the claim (5.3) is true for almost every $Y \in \mathbb{R}$ at $T = 0$. Then, (5.3) remains true for all times $T \in \mathbb{R}$ and a.e. $Y \in \mathbb{R}$. Then, for any $Y_1 \neq Y_2$ (without loss of generality, we assume $Y_1 < Y_2$), if $x(t^*, Y_1) = x(t^*, Y_2)$, then, from the monotonicity of $x(t, Y)$ on Y , for every $Y \in [Y_1, Y_2]$, we have $x(t^*, Y) = x(t^*, Y_1)$. And, from (5.3) we get

$$0 = x(t^*, Y_1) - x(t^*, Y_2) = \int_{Y_1}^{Y_2} x_Y(t^*, Y) dY = \int_{Y_1}^{Y_2} \cos^2 \frac{v(t^*, Y)}{2} \cdot \xi(t^*, Y) dY.$$

Then, $\cos \frac{v(t^*, Y)}{2} \equiv 0$ for every $Y \in [Y_1, Y_2]$. Inject this into (4.2).

$$\begin{aligned} u(t^*, Y_1) - u(t^*, Y_2) &= \int_{Y_1}^{Y_2} u_Y(t^*, Y) dY \\ &= \frac{1}{2} \int_{Y_1}^{Y_2} \xi(t^*, Y) \sin \frac{v(t^*, Y)}{2} \cos \frac{v(t^*, Y)}{2} dY = 0. \end{aligned}$$

This proves that the map $(t, x) \mapsto u(t(T), x(T, Y))$ at (5.2) is well defined for all $(t, x) \in \mathbb{R}^2$.

Secondly, we prove the regularity of $u(t, x)$ and energy equation. From (4.4), we see that for t in any bounded interval,

$$\begin{aligned} &\int_{\mathbb{R}} (u^2(t, x) + u_x^2(t, x)) dx \\ &= \int_{\{\mathbb{R} \cap \cos v \neq -1\}} [u^2(T, Y) \cos^2 \frac{v(T, Y)}{2} + \sin^2 \frac{v(T, Y)}{2}] \xi(T, Y) dY \quad (5.4) \\ &\leq D_0^2. \end{aligned}$$

Now, applying the Sobolev inequality: $\|u\|_{C^{0, \gamma}} \leq C\|u\|_{H^1}$, where $\gamma = 1 - \frac{1}{2}$. From (5.4), we get $\|u\|_{C^{0, \frac{1}{2}}} \leq C\|u\|_{H^1} \leq CD_0$, which implies u of x is Hölder continuous with index $\frac{1}{2}$. On the other hand, it follows from the first identity in (2.6), (4.6) and (4.7) that $\|u_t\|_{L^\infty} \leq C(\|P_x\|_{L^\infty} + \|Q\|_{L^\infty}) \leq C(D_0)$. Then, the map $t \mapsto u(t, x(t))$ is Lipschitz continuous along every characteristic curve $t \mapsto x(t)$. Therefore, $u = u(t, x)$ is Hölder continuous on the any bounded interval of times.

Thirdly, we prove that the $L^2(\mathbb{R})$ -norm of $u(t)$ is Lipschitz continuous with respect to t on any bounded interval. Denote $[\tau, \tau + h]$ to be any small interval and τ in any bounded interval. For a given point (τ, \bar{x}) , we choose the characteristic $t \mapsto x(t, Y) : \{T \rightarrow x(T, Y)\}$ passes through the point (τ, \bar{x}) , i.e. $x(\tau) = \bar{x}$. Since the characteristic speed u satisfies $\|u\|_{L^\infty} \leq C\|u\|_{H^1} \leq CD_0$, we have the following estimate.

$$\begin{aligned} &|u(\tau + h, \bar{x}) - u(\tau, \bar{x})| \\ &\leq |u(\tau + h, \bar{x}) - u(\tau + h, x(\tau + h, Y))| \\ &\quad + |u(\tau + h, x(\tau + h, Y)) - u(\tau, x(\tau, Y))| \\ &\leq \sup_{|y - \bar{x}| \leq CD_0 h} |u(\tau + h, y) - u(\tau + h, \bar{x})| + \int_\tau^{\tau + h} |P_x(t, Y)| + k|Q(t, Y)| dt \end{aligned}$$

Then, we use $(\int_a^b f(x)dx)^2 \leq [\sqrt{b-a}\|f\|_{L^2}]^2$ for $a < b$, and deduce that

$$\begin{aligned}
& \int_{\mathbb{R}} |u(\tau + h, \bar{x}) - u(\tau, \bar{x})|^2 dx \\
& \leq C \int_{\mathbb{R}} (\int_{\bar{x}-CD_0h}^{\bar{x}+CD_0h} |u_x(\tau + h, y)|^2 dy) dx \\
& \quad + C \int_{\mathbb{R}} (\int_{\tau}^{\tau+h} |P_x(t, Y)| + k|Q(t, Y)| dt)^2 \cdot \xi(\tau, Y) dY \\
& \leq C \int_{\mathbb{R}} 2D_0h \int_{\bar{x}-CD_0h}^{\bar{x}+CD_0h} |u_x(\tau + h, y)|^2 dy dx \\
& \quad + C \int_{\mathbb{R}} (h \int_{\tau}^{\tau+h} |P_x(t, Y)|^2 + k|Q(t, Y)|^2 dt) \|\xi(\tau)\|_{L^\infty} dY \\
& \leq 4CD_0^2h^2\|u_x(\tau)\|_{L^2}^2 + Ch^2\|\xi(\tau)\|_{L^\infty}(\|P_x\|_{L^2}^2 + k\|Q\|_{L^2}^2) \\
& \leq C(D_0)h^2.
\end{aligned} \tag{5.5}$$

Then, This implies that the map $t \mapsto u(t)$ is Lipchitz continuous, in terms of the x -variable in $L^2(\mathbb{R})$.

Fourthly, we prove that the function u provides a weak solution of (1.2). Denote $\Gamma := \{(t, x) | t \in \mathbb{R}, x \in \mathbb{R}\}$. For any test function $\phi(t, x) \in C_c^1(\Gamma)$, the first equation of (2.6) has the following weak form.

$$\begin{aligned}
0 &= \int_{\Gamma} \{u_{TY} + (P_x)_Y - kQ_Y\} \phi dY dT \\
&= \int_{\Gamma} \{u_{TY} \phi + [-u^2 \cos^2 \frac{v}{2} - \frac{1}{2} \sin^2 \frac{v}{2} + (P - kQ_x) \cos^2 \frac{v}{2}] \xi \phi\} dY dT \\
&= \int_{\Gamma} \{-u_Y \phi_T + [-u^2 \cos^2 \frac{v}{2} - \frac{1}{2} \sin^2 \frac{v}{2} + (P - kQ_x) \cos^2 \frac{v}{2}] \xi \phi\} dY dT \\
&= \int_{\Gamma} \{-u_x \phi_T \xi \cos^2 \frac{v}{2} + [-u^2 \cos^2 \frac{v}{2} - \frac{1}{2} \sin^2 \frac{v}{2} + (P - kQ_x) \cos^2 \frac{v}{2}] \xi \phi\} dY dT \\
&= \int_{\Gamma} \{-u_x(\phi_t + u\phi_x) + [-u^2 - \frac{1}{2}u_x^2 + P - kQ_x] \phi\} dx dt,
\end{aligned}$$

which implies that the weak form (1.8) is true. Now, we introduce the Radon measures $\{\mu_{(t)}, t \in \mathbb{R}\}$: for any Lebesgue measurable set $\{x \in \mathcal{A}\}$ in \mathbb{R} . Supposing the corresponding pre-image set of the transformation is $\{Y \in \mathcal{G}(\mathcal{A})\}$, we have

$$\mu_{(t)}(\mathcal{A}) = \int_{\mathcal{G}(\mathcal{A})} \xi \sin^2 \frac{v}{2}(t, Y) dY.$$

For every $t \in \mathbb{R} \setminus N$, the absolutely continuous part of $\mu_{(t)}$ w.r.t. Lebesgue measure has density $u_x^2(t, \cdot)$ by (5.3). It follows from (2.6) that for any test function $\phi(t, x) \in C_c^1(\Gamma)$,

$$\begin{aligned}
-\int_{\mathbb{R}^+} \{\int_{\mathbb{R}} (\phi_t + u\phi_x) d\mu_{(t)}\} dt &= -\int_{\Gamma} \phi_T \xi \sin^2 \frac{v}{2} dY dT \\
&= \int_{\Gamma} \phi (\xi \sin^2 \frac{v}{2})_T dY dT \\
&= \int_{\Gamma} 2\phi \xi (u^2 - P + kQ_x) \cos \frac{v}{2} \sin \frac{v}{2} dY dT \\
&= \int_{\Gamma} 2u_x(u^2 - P + kQ_x) \phi dx dt.
\end{aligned}$$

Then, (1.9) is true due to $\text{meas}(N) = 0$.

Finally, let $u_{0,n}$ be a sequence of initial data converging to u_0 in $H^1(\mathbb{R})$. From (2.1) and (2.7), at time $t = 0$ this implies

$$\sup_{Y \in \mathbb{R}} |x_n(0, Y) - x(0, Y)| \rightarrow 0, \quad \sup_{Y \in \mathbb{R}} |u_n(0, Y) - u(0, Y)| \rightarrow 0,$$

Meanwhile, $\|v_n(0, \cdot) - v(0, \cdot)\|_{L^2} \rightarrow 0$. This implies that $u_n(T, Y) \rightarrow u(T, Y)$, uniformly for T, Y in bounded sets. Returning to the original coordinates, this yields the convergence

$$x_n(T, Y) \rightarrow x(T, Y), \quad u_n(t, x) \rightarrow u(t, x),$$

uniformly on bounded sets, because all functions u, u_n are Hölder continuous. This completes the proof. \square

6 Uniqueness of the Global Weak Solution

We shall give the proof of Theorem 1.2 in the case $t \geq 0$, and the case $t < 0$ can be handled by the similar argument.

Step 1. We define $x(t, \beta)$ to be the unique point x such that

$$x(t, \beta) + \mu_{(t)}\{(-\infty, x)\} \leq \beta \leq x(t, \beta) + \mu_{(t)}\{(-\infty, x]\}. \quad (6.1)$$

Recalling that at every time, $\mu_{(t)}$ is absolutely continuous with density u_x^2 w.r.t. Lebesgue measure, the above definition gives

$$\beta := x(t, \beta) + \int_{-\infty}^{x(t, \beta)} u_x^2(t, z) dz = x(t, \beta) + \mu_{(t)}\{(-\infty, x(t, \beta))\}. \quad (6.2)$$

We study the Lipschitz continuity of x and u as functions of t, β .

Lemma 6.1. *Let $u = u(t, x)$ be the weak solution of (1.2) satisfying (1.9). Then, for every $t \geq 0$,*

(i) $\beta \mapsto x(t, \beta)$ and $\beta \mapsto u(t, \beta) := u(t, x(t, \beta))$ implicitly defined by (6.2) are Lipschitz continuous with the constant 1,

(ii) $t \mapsto x(t, \beta)$ is Lipschitz continuous with a constant relaying on $\|u_0\|_{H^1}$.

Proof. (i) From the definition of β in (6.2), we remark that for any time $t \geq 0$, $x \mapsto \beta(t, x)$ is right continuous and strictly increasing. Then, its

inverse $\beta \mapsto x(t, \beta)$ is well-defined, and is also continuous and nondecreasing. For any $\beta_1 < \beta_2$, we deduce that

$$\begin{aligned}\beta_2 - \beta_1 &= x(t, \beta_2) - x(t, \beta_1) + \int_{-\infty}^{x(t, \beta_1)} u_x^2(t, z) dz - \int_{-\infty}^{x(t, \beta_2)} u_x^2(t, z) dz \\ &\geq x(t, \beta_2) - x(t, \beta_1) + \mu_{(t)}\{(x(t, \beta_1), x(t, \beta_2))\}.\end{aligned}\tag{6.3}$$

Hence, we get $x(t, \beta_2) - x(t, \beta_1) \leq \beta_2 - \beta_1$, and the map $\beta \mapsto x(t, \beta)$ is Lipchitz continuous with the constant 1. For $\beta \mapsto u(t, \beta)$, from $|u(x_1) - u(x_2)| \leq \int_{x_1}^{x_2} |u_x| dx \leq \frac{1}{2}(|x_2 - x_1| + \int_{x_1}^{x_2} u_x^2 dx)$ we see that for any $\beta_1 < \beta_2$

$$|u(t, x(t, \beta_2)) - u(t, x(t, \beta_1))| \leq \frac{1}{2}[x(t, \beta_2) - x(t, \beta_1) + \mu_{(t)}\{(x(t, \beta_1), x(t, \beta_2))\}].$$

From (6.3), $\beta \mapsto u(t, \beta)$ is Lipchitz continuous with constant $\frac{1}{2} < 1$.

(ii) From the Sobolev embedding inequality, we have $\|u\|_{L^\infty} \leq C\|u\|_{H^1} := C_\infty$. Assume $x(t, \beta) = y$. We remark that the family of measures $\mu_{(t)}$ satisfies the balance law (1.9), where for each t , the source term $2(u^2 - P + kQ_x)u_x$ in (1.9) has the following estimate by the Hölder inequality.

$$\|2(u^2 - P + kQ_x)u_x\|_{L^1} \leq 2\|u_x\|_{L^2}(\|P\|_{L^2} + k\|Q_x\|_{L^2} + \|u^2\|_{L^2}) \leq C_0,$$

where C_0 depends only on $\|u\|_{H^1}$. Thus, for any $t > \tau$, from (1.9),

$$\begin{aligned}\mu_{(t)}\{(-\infty, y - C_\infty(t - \tau))\} &\leq \mu_{(\tau)}\{(-\infty, y)\} + \int_\tau^t \|2(u^2 - P + kQ_x)u_x\|_{L^1} dt \\ &\leq \mu_{(\tau)}\{(-\infty, y)\} + C_0(t - \tau).\end{aligned}$$

Let $y^-(t) = y - (C_\infty + C_0)(t - \tau)$. Then, we have

$$\begin{aligned}y^-(t) + \mu_{(t)}\{(-\infty, y^-(t))\} &\leq y - (C_\infty + C_0)(t - \tau) + \mu_{(t)}\{(-\infty, y)\} + C_0(t - \tau) \\ &\leq y - \mu_{(\tau)}\{(-\infty, y)\} \leq \beta.\end{aligned}$$

This implies that $x(t, \beta) \geq y^-(t)$ for all $t > \tau$. And we can obtain $x(t, \beta) \leq y^+(t) := y + (C_\infty + C_0)(t - \tau)$ by using the similar argument. This completes the proof of the uniformly Lipchitz continuity of the mapping $t \mapsto x(t, \beta)$. \square

Step 2. The next lemma shows that characteristics can be uniquely determined by an integral equation combining the characteristic equation and balance law of u_x^2 , which is crucial to study the uniqueness of the conservative solution of Eq.(1.2).

Lemma 6.2. *Let $u = u(t, x)$ be the weak solution of Eq.(1.2) satisfying (1.9). Then, for any $x_0 \in \mathbb{R}$ there exists a unique Lipschitz continuous map $t \mapsto x(t)$ which satisfies*

$$\frac{d}{dt}x(t) = u(t, x(t)), \quad x(0) = x_0 \quad (6.4)$$

and

$$\frac{d}{dt} \int_{-\infty}^{x(t)} u_x^2(t, x) dx = \int_{-\infty}^{x(t)} [2(u^2 - P + kQ_x)u_x] (t, x) dx, \quad x(0) = x_0, \quad (6.5)$$

for a.e. $t \geq 0$. Furthermore, for any $0 \leq \tau \leq t$, we have

$$u(t, x(t)) - u(\tau, x(\tau)) = - \int_{\tau}^t (P_x - kQ)(s, x(s)) ds. \quad (6.6)$$

Proof. Firstly, by the adapted coordinates (t, β) , we write the characteristic starting at x_0 in the form $t \mapsto x(t) = x(t, \beta(t))$, where $\beta(\cdot)$ is a map to be determined. Sum up (6.4) and (6.5) and integrate w.r.t. time. We get

$$\beta(t) = \beta_0 + \int_0^t G(s, \beta(s)) ds. \quad (6.7)$$

where $\beta = x(t) + \int_{-\infty}^{x(t)} u_x^2(t, x) dx$, $\beta_0 = x_0 + \int_{-\infty}^{x_0} (u_0)_x^2(x) dx$ and

$$G(t, \beta) = \int_{-\infty}^{x(s)} [u_x + 2(u^2 - P + kQ_x)u_x](s, x) dx. \quad (6.8)$$

For each fixed $t \geq 0$, since the maps $x \mapsto u(t, x)$, $x \mapsto P(t, x)$ and $x \mapsto Q_x(t, x)$ are both in $H^1(\mathbb{R})$, the function $\beta \mapsto G(t, \beta)$ defined at (6.8) is uniformly bounded and absolutely continuous. Moreover,

$$G_{\beta} = [u_x + 2(u^2 - P + kQ_x)u_x]_{\beta} = \frac{u_x + 2(u^2 - P + kQ_x)u_x}{1 + u_x^2} \in [-C, C]$$

for some constant C depending only on the H^1 -norm of u . Hence the function G in (6.8) is Lipschitz continuous w.r.t. β . We can apply the ODE's theory in the Banach space of all continuous functions $\beta : \mathbb{R}^+ \mapsto \mathbb{R}$ with weighted norm $\|\beta\| := \sup_{t \geq 0} e^{-2Ct} |\beta(t)|$. Let $[\Phi\beta](t) := \beta_0 + \int_0^t G(\tau, \beta(\tau)) d\tau$.

Assume $\|\beta - \beta_0\| = \delta > 0$. we have $|\beta_1(\tau) - \beta_2(\tau)| \leq \delta e^{2C\tau}$ for all $\tau \geq 0$. By the Lipchitz continuity of G ,

$$|[\Phi\beta_1](t) - [\Phi\beta_2](t)| \leq C \int_0^t |\beta_1(\tau) - \beta_2(\tau)| d\tau \leq \frac{\delta}{2} e^{2Ct}.$$

Then, $[\Phi\beta]$ is a strict contraction. (6.8) has a unique solution $t \mapsto \beta(t)$, and the corresponding function $t \mapsto x(t, \beta(t))$ satisfies (6.4) and (6.5).

Secondly, from the integral equation (6.7), we can determine a Lipschitz continuous characteristic $x(t)$ of (6.4). By the previous construction, the map $t \mapsto x(t) := x(t, \beta(t))$ provides the unique solution to (6.7). Being the composition of two Lipschitz functions, the map $t \mapsto x(t, \beta(t))$ is also Lipschitz continuous. To prove that it satisfies the ODE for the characteristics of (6.4), it suffices to show that (6.4) holds at each time $\tau > 0$ such that

- (i) $x(\cdot)$ is differentiable at $t = \tau$,
- (ii) the measure $\mu_{(\tau)}$ is absolutely continuous.

Assume, on the contrary, that (i) and (ii) hold but $\frac{d}{dt}x(\tau) \neq u(\tau, x(\tau))$. Let

$$\frac{d}{dt}x(\tau) = u(\tau, x(\tau)) + 2\varepsilon_0$$

for some $\varepsilon_0 > 0$ (the case $\varepsilon_0 < 0$ can be handled similarly). To derive a contradiction we see that, for all $t \in (\tau, \tau + \delta]$, with $\delta > 0$ small enough

$$x^+(t) := x(\tau) + (t - \tau)[u(\tau, x(\tau)) + \varepsilon_0] < x(t). \quad (6.9)$$

We also observe that if ϕ is Lipschitz continuous with compact support then (1.9) is still true. For any $\epsilon > 0$ small, we will use the test functions.

$$\begin{aligned} \rho^\epsilon(s, y) &:= \begin{cases} 0 & \text{if } y \leq -\epsilon^{-1}, \\ (y + \epsilon^{-1}) & \text{if } -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\ 1 & \text{if } 1 - \epsilon^{-1} \leq y \leq x^+(s), \\ 1 - \epsilon^{-1}(y - x(s)) & \text{if } x^+(s) \leq y \leq x^+(s) + \epsilon, \\ 0 & \text{if } y \geq x^+(s) + \epsilon, \end{cases} \\ \chi^\epsilon(s) &:= \begin{cases} 0 & \text{if } s \leq \tau - \epsilon, \\ \epsilon^{-1}(s - \tau + \epsilon) & \text{if } \tau - \epsilon \leq s \leq \tau, \\ 1 & \text{if } \tau \leq s \leq t, \\ 1 - \epsilon^{-1}(s - t) & \text{if } t \leq s < t + \epsilon, \\ 0 & \text{if } s \geq t + \epsilon. \end{cases} \end{aligned} \quad (6.10)$$

Let $\varphi^\epsilon(s, y) := \min\{\rho^\epsilon(s, y), \chi^\epsilon(s)\}$. Use φ^ϵ as test function in (1.9).

$$\int_{\Gamma} [u_x^2 \varphi_t^\epsilon + u u_x^2 \varphi_x^\epsilon + 2(u^2 - P + kQ_x) u_x \varphi^\epsilon] dx dt = 0. \quad (6.11)$$

For $s \in [\tau + \epsilon, t - \epsilon]$, we get $\varphi_x^\epsilon \leq 0$ and $u(s, x) < u(\tau, x(\tau)) + \epsilon_0$ by the Hölder continuity of u . Then, $\varphi_t^\epsilon + u(s, x) \varphi_x^\epsilon \geq \varphi_t^\epsilon + [u(\tau, x(\tau)) + \epsilon_0] \varphi_x^\epsilon = 0$.

Thus, $\lim_{\epsilon \rightarrow 0} \int_{\tau}^t \int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon} u_x^2(\varphi_t^\epsilon + u\varphi_x^\epsilon)(s, x) dx ds \geq 0$ as t is sufficiently close to τ . Since the family of measure $\mu_{(t)}$ depends continuously on t in the topology of weak convergence, by taking $\epsilon \rightarrow 0$ in (6.11), we have $\tau, t \notin N$

$$\begin{aligned}
& \int_{-\infty}^{x^+(t)} u_x^2(t, x) dx \\
&= \int_{-\infty}^{x(\tau)} u_x^2(\tau, x) dx + \int_{\tau}^t \int_{-\infty}^{x^+(s)} 2(u^2 - P + kQ_x)u_x dx ds \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_{\tau}^t \int_{x^+(s)-\epsilon}^{x^+(s)+\epsilon} u_x^2(\varphi_t^\epsilon + u\varphi_x^\epsilon) dx ds \\
&\geq \int_{-\infty}^{x(\tau)} u_x^2(\tau, x) dx + \int_{\tau}^t \int_{-\infty}^{x(s)} 2(u^2 - P + kQ_x)u_x dx ds + o_1(t - \tau),
\end{aligned} \tag{6.12}$$

where $\frac{o_1(t-\tau)}{t-\tau} \rightarrow 0$ is a higher order infinitesimal satisfying

$$\begin{aligned}
|o_1(t - \tau)| &= \left| \int_{\tau}^t \int_{x^+(s)}^{x(s)} 2(u^2 - P + kQ_x)u_x dx ds \right| \\
&\leq \|2(u^2 - P + kQ_x)\|_{L^\infty} \int_{\tau}^t |x(s) - x^+(s)|^{\frac{1}{2}} \|u_x(s, \cdot)\|_{L^2} ds \\
&\leq C(t - \tau)^{\frac{3}{2}} \rightarrow 0 \quad \text{as } t \rightarrow \tau.
\end{aligned}$$

For every $t > \tau$ with $t \notin N$, when t is sufficiently close to τ , by injecting (6.9) and (6.12) into $\beta(t)$, we deduce that

$$\begin{aligned}
\beta(t) &> x(\tau) + (t - \tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x^+(t)} u_x^2(t, x) dx \\
&\geq x(\tau) + (t - \tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x(\tau)} u_x^2(\tau, x) dx \\
&\quad + \int_{\tau}^t \int_{-\infty}^{x(s)} 2(u^2 - P + kQ_x)u_x dx ds + o_1(t - \tau).
\end{aligned} \tag{6.13}$$

On the other hand, from (6.7) and (6.8), a linear approximation yields

$$\beta(t) = \beta(\tau) + (t - \tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x(\tau)} 2(u^2 - P + kQ_x)u_x dx + o_2(t - \tau), \tag{6.14}$$

with $o_2(t - \tau) := \int_{\tau}^t \int_{x^+(s)}^{x(s)} [u_x + 2(u^2 - P + kQ_x)u_x] dx ds$, and $\lim_{t \rightarrow \tau} \frac{o_2(t - \tau)}{t - \tau} = 0$.

By combining (6.14) and (6.13), we see that

$$\begin{aligned}
& x(\tau) + (t - \tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x(\tau)} u_x^2(\tau, x) dx \\
& \quad + \int_{\tau}^t \int_{-\infty}^{x(s)} 2(u^2 - P + kQ_x)u_x dx ds + o_1(t - \tau) \\
& \leq \beta(\tau) + (t - \tau)[u(\tau, x(\tau)) + \varepsilon_0] + \int_{-\infty}^{x(\tau)} 2(u^2 - P + kQ_x)u_x dx + o_2(t - \tau).
\end{aligned}$$

Subtracting common terms and dividing both sides by $t - \tau$ and letting $t \rightarrow \tau$, we get $\varepsilon_0 \leq 0$, which is a contradiction. Then, (6.4) must hold.

Thirdly, we prove (6.6). In (1.8), let $\phi = \varphi_x$ and $\varphi \in C_c^\infty$. Since the map $x \mapsto u(t, x)$ is absolutely continuous, we can integrate by parts w.r.t. x .

$$\int_{\Gamma} [u_x \varphi_t + uu_x \varphi_x + (P_x - kQ) \varphi_x] dx dt + \int_{\mathbb{R}} (u_0)_x(x) \varphi(0, x) dx = 0. \tag{6.15}$$

By an approximation argument, (6.15) remains valid for any test function φ which is Lipschitz continuous with compact support. For any $\epsilon > 0$ sufficiently small, we thus consider the function:

$$\varrho^\epsilon(s, y) := \begin{cases} 0 & \text{if } y \leq -\epsilon^{-1}, \\ y + \epsilon^{-1} & \text{if } -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\ 1 & \text{if } 1 - \epsilon^{-1} \leq y \leq x(s), \\ 1 - \epsilon^{-1}(y - x(s)) & \text{if } x(s) \leq y \leq x(s) + \epsilon, \\ 0 & \text{if } y \geq x(s) + \epsilon. \end{cases}$$

Then, we define $\psi^\epsilon(s, y) := \min\{\varrho^\epsilon(s, y), \chi^\epsilon(s)\}$, where $\chi^\epsilon(s)$ as in (6.10). Take $\varphi = \psi^\epsilon$ in (6.15) and let $\epsilon \rightarrow 0$. From the continuity of $(P_x - kQ)$,

$$\begin{aligned} \int_{-\infty}^{x(t)} u_x(t, x) dx &= \int_{-\infty}^{x(\tau)} u_x(\tau, x) dx - \int_{\tau}^t (P_x - kQ)(s, x(s)) ds \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{\tau-\epsilon}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} u_x(\psi_t^\epsilon + u\psi_x^\epsilon) dx ds. \end{aligned} \quad (6.16)$$

For every time $s \in [\tau - \epsilon, t + \epsilon]$ by construction, we see that

$$\psi_x^\epsilon(s, y) = \epsilon^{-1}, \psi_t^\epsilon(s, y) + u(s, x(s))\psi_x^\epsilon(s, y) = 0 \quad \text{for } x(s) < y < x(s) + \epsilon.$$

This implies

$$\begin{aligned} \int_{x(s)}^{x(s)+\epsilon} |\psi_t^\epsilon(s, y) + u(s, y)\psi_x^\epsilon(s, y)|^2 dy &= \frac{1}{\epsilon^2} \int_{x(s)}^{x(s)+\epsilon} |u(s, x(s)) - u(s, y)|^2 dy \\ &\leq \frac{1}{\epsilon} \left(\max_{x(s) \leq y \leq x(s)+\epsilon} |u(s, y) - u(s, x(s))| \right)^2 \leq \frac{1}{\epsilon} \left(\int_{x(s)}^{x(s)+\epsilon} |u_x(s, y)| dy \right)^2 \\ &\leq \frac{1}{\epsilon} (\epsilon \|u_x(s)\|_{L^2})^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

Then, we have

$$\begin{aligned} \left| \int_{\tau-\epsilon}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} u_x(\psi_t^\epsilon + u\psi_x^\epsilon) dx ds \right| &\leq C \|u_x\|_{L^2} \left(\int_{x(s)}^{x(s)+\epsilon} (\psi_t^\epsilon + u\psi_x^\epsilon)^2 dx \right)^{\frac{1}{2}} \\ &\leq C \epsilon^{\frac{1}{2}} \|u(s)\|_{H^1}^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned} \quad (6.17)$$

It follows from (6.16) and (6.17) that (6.6) is true.

Finally, we prove the uniqueness of $x(t)$. Assume that there exist two different $x_1(t)$ and $x_2(t)$, which satisfy (6.4) and (6.5). Now, choosing the measurable functions β_1 and β_2 such that $x_1(t) = x(t, \beta_1(t))$ and $x_2(t) = x(t, \beta_2(t))$. Then, $\beta_1(\cdot)$ and $\beta_2(\cdot)$ satisfy (6.7) with the same initial data $x(0) = x_0$. This contradicts with the uniqueness of β . \square

Step 3. We give some additional properties of β and u , as follows.

Lemma 6.3. *If $u = u(t, x)$ is the weak solution of Eq. (1.2) satisfying (1.9). Then,*

- (i) the mapping $(t, \beta) \mapsto u(t, \beta) := u(t, x(t, \beta))$ is Lipchitz continuous with a constant depending only on the norm $\|u_0\|_{H^1}$,
- (ii) denote $t \mapsto \beta(t; \tau, \beta_0)$ be the solution to the integral equation

$$\beta(t) = \beta_0 + \int_{\tau}^t G(\tau, \beta(\tau)) d\tau. \quad (6.18)$$

We deduce that there exists a constant C such that for any $\beta_{1,0}, \beta_{2,0}$ and any $t, \tau \geq 0$ the corresponding solutions satisfy

$$|\beta(t; \tau, \beta_{1,0}) - \beta(t; \tau, \beta_{2,0})| \leq e^{C|t-\tau|} |\beta_{1,0} - \beta_{2,0}|. \quad (6.19)$$

Proof. (i) It follows from (6.2), (6.6), and (6.7) that

$$\begin{aligned} & |u(t, x(t, \beta_0)) - u(\tau, \beta_0)| \\ & \leq |u(t, x(t, \beta_0)) - u(t, x(t, \beta(t)))| + |u(t, x(t, \beta(t))) - u(\tau, x(\tau, \beta(\tau)))| \\ & \leq \frac{1}{2} |\beta(t) - \beta_0| + (t - \tau) \|P_x - kQ\|_{L^\infty} \\ & \leq C(t - \tau), \end{aligned}$$

where $C := \frac{1}{2} \|G\|_{L^\infty} + \|P_x\|_{L^\infty} + k\|Q\|_{L^\infty} > 0$ depends only on $\|u_0\|_{H^1}$.

(ii) It follows from the Lipchitz continuity of G that

$$|\beta(t; \tau, \beta_{1,0}) - \beta(t; \tau, \beta_{2,0})| \leq |\beta_{1,0} - \beta_{2,0}| + C \int_{\tau}^t |\beta(s; \tau, \beta_{1,0}) - \beta(s; \tau, \beta_{2,0})| ds.$$

Then, we can obtain (6.19) by the Gronwall inequality. \square

Step 4. We give the proof of Theorem 1.2.

Proof. Firstly, by Lemma 6.1 and Lemma 6.3, the map $(t, \beta) \mapsto (x, u)(t, \beta)$ is Lipschitz continuous. An entirely similar argument shows that the maps $\beta \mapsto G(t, \beta) := G(t, x(t, \beta))$ and $\beta \mapsto P(t, \beta) := P(t, x(t, \beta))$, $\beta \mapsto P_x(t, \beta) := P_x(t, x(t, \beta))$, $\beta \mapsto Q(t, \beta) := Q(t, x(t, \beta))$ and $\beta \mapsto Q_x(t, \beta) := Q_x(t, x(t, \beta))$ are also Lipschitz continuous. By Rademacher's theorem in [19], the partial derivatives $x_t, x_\beta, u_t, u_\beta, G_\beta, P_\beta, Q_\beta, (P_x)_\beta$ and $(Q_x)_\beta$ exist almost everywhere. Moreover, a.e. point (t, β) is a Lebesgue point for these derivatives. Calling $t \mapsto \beta(t, \beta_0)$ the unique solution to the integral equation (6.7), by Lemma 6.2 for a.e. β_0 the following holds.

(GC) For a.e. $t > 0$, the point $(t, \beta(t, \beta_0))$ is a Lebesgue point for the partial derivatives $x_t, x_\beta, u_t, u_\beta, G_\beta, P_\beta, Q_\beta, (P_x)_\beta, (Q_x)_\beta$. Moreover, $x_\beta(t, \beta(t, \beta_0)) > 0$ for a.e. $t > 0$.

If (GC) holds, then we say that $t \mapsto \beta(t, \beta_0)$ is a **good characteristic**. We seek an ODE describing how the quantities u_β and x_β vary along a good characteristic. As in Lemma 6.3, we denote by $t \mapsto Z(t) := \beta(t; \tau, \beta_0)$ to the solution of (6.18). If $\tau, t \notin N$, assuming that $Z(t)$ is a good characteristic and differentiating (6.18) w.r.t. β_0 , we find

$$\frac{\partial}{\partial \beta_0} Z(t) = 1 + \int_\tau^t G_\beta(s, Z(s)) \cdot \frac{\partial}{\partial \beta_0} Z(s) ds \quad (6.20)$$

Next, differentiate $x(t, Z(t)) = x(\tau, \beta_0) + \int_\tau^t u(s, x(s, Z(s))) ds$ w.r.t. β_0 .

$$x_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t) = x_\beta(\tau, \beta_0) + \int_\tau^t u_\beta(s, Z(s)) \cdot \frac{\partial}{\partial \beta_0} Z(s) ds. \quad (6.21)$$

By differentiating w.r.t. β_0 the identity (6.6), we obtain

$$u_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t) = u_\beta(\tau, \beta_0) - \int_\tau^t (P_x - kQ)_\beta(s, Z(s)) \cdot \frac{\partial}{\partial \beta_0} Z(s) ds. \quad (6.22)$$

Combining (6.20)–(6.22), we thus obtain the system of ODEs

$$\begin{cases} \frac{d}{dt} [\frac{\partial}{\partial \beta_0} Z(t)] = G_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t), \\ \frac{d}{dt} [x_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t)] = u_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t), \\ \frac{d}{dt} [u_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t)] = -(P_x - kQ)_\beta(t, Z(t)) \cdot \frac{\partial}{\partial \beta_0} Z(t). \end{cases} \quad (6.23)$$

In particular, the quantities within square brackets on the left hand sides of (6.23) are absolutely continuous. Recall the fact $P_{xx} = P - (\frac{1}{2}u_x^2 + u^2)$ and $u_x^2 = \frac{1-x_\beta}{x_\beta}$. From (6.23), along a good characteristic we obtain

$$\begin{cases} \frac{d}{dt} x_\beta + G_\beta x_\beta = u_\beta, \\ \frac{d}{dt} u_\beta + G_\beta u_\beta = (u^2 - P + kQ_x) x_\beta + \frac{1}{2} u_x^2 \frac{1}{1+u_x^2}, \end{cases} \quad (6.24)$$

where the first equation is obtained by the first two equations in (6.23) and the second equation is obtained by the first and third equations in (6.23)

Secondly, we go back to the original (t, x) coordinates and derive an evolution equation for the partial derivative u_x along a "good" characteristic curve. Fix a point (τ, x_0) with $\tau \notin N$. Assume that x_0 is a Lebesgue point for the map $x \mapsto u_x(\tau, x)$. Let β_0 be such that $x_0 = x(\tau, \beta_0)$ and assume that $t \mapsto Z(t)$ is a *good characteristic*, so that (GC) holds. We observe that $u_x^2(\tau, x) = \frac{1}{x_\beta(\tau, \beta_0)} - 1 \geq 0$, which implies $x_\beta(\tau, \beta_0) > 0$. As long as $x_\beta > 0$, along the characteristic through (τ, x_0) the partial derivative u_x can

be computed as $u_x(t, x(t, Z(t))) = \frac{u_\beta(t, Z(t))}{x_\beta(t, Z(t))}$. Using (6.21)-(6.22) describing the evolution of u_β and x_β , we conclude that the map $t \mapsto u_x(t, x(t, Z(t)))$ is absolutely continuous (as long as $x_\beta \neq 0$) and satisfies

$$\frac{d}{dt} u_x(t, x(t, Z(t))) = \frac{d}{dt} \left(\frac{u_\beta(t, Z(t))}{x_\beta(t, Z(t))} \right) = u^2 - P + kQ_x - \frac{1}{2} \frac{u_\beta^2}{x_\beta^2}.$$

We remark $u_\beta^2 = x_\beta^2 u_x^2 = x_\beta(1 - x_\beta)$, and as long as $x_\beta > 0$ this implies

$$\frac{d}{dt} \arctan u_x(t, x(t, Z(t))) = (u^2 - P + kQ_x) x_\beta - \frac{1}{2} (1 - x_\beta). \quad (6.25)$$

Define the function

$$v := \begin{cases} 2 \arctan u_x & \text{if } 0 < x_\beta \leq 1, \\ \pi & \text{if } x_\beta = 0. \end{cases}$$

Then, we see that

$$x_\beta = \frac{1}{1 + u_x^2} = \cos^2 \frac{v}{2}, \quad 1 - x_\beta = \frac{u_x^2}{1 + u_x^2} = \sin^2 \frac{v}{2}. \quad (6.26)$$

In the following, v will be regarded as a map taking values in the unit circle $\mathcal{S} := [-\pi, \pi]$ with endpoints identified. We claim that, along each good characteristic, the map $t \mapsto v(t) := v(t, x(t, \beta(t; \tau, \beta_0)))$ is absolutely continuous and satisfies

$$\frac{d}{dt} v(t) = 2(u^2 - P + kQ_x) \cos^2 \frac{v}{2} - \sin^2 \frac{v}{2}. \quad (6.27)$$

Indeed, denote by $x_\beta(t)$, $u_\beta(t)$ and $u_x(t) = u_\beta(t)/x_\beta(t)$ the values of x_β , u_β , and u_x along this particular characteristic. By (GC) we have $x_\beta(t) > 0$ for a.e. $t > 0$.

If τ is any time where $x_\beta(\tau) > 0$, we can find a neighborhood $I = [\tau - \delta, \tau + \delta]$ such that $x_\beta(t) > 0$ on I . By (6.25) and (6.26), $v = 2 \arctan(u_\beta/x_\beta)$ is absolutely continuous restricted to I and satisfies (6.27). To prove our claim, it thus remains to show that $t \mapsto v(t)$ is continuous on the null set N of times where $x_\beta(t) = 0$. Suppose $x_\beta(0) = 0$. From the fact that the identity $u_x^2(t) = \frac{1 - x_\beta(t)}{x_\beta(t)}$ holds as long as $x_\beta > 0$, it is clear that $u_x^2 \rightarrow +\infty$ as $t \rightarrow 0$ and $x_\beta(t) \rightarrow 0$. This implies $v(t) = 2 \arctan u_x(t) \rightarrow \pm\pi$. Since we identify the points $\pm\pi$, we get the continuity of v for all $t \geq 0$, proving our claim.

Thirdly, let $u = u(t, x)$ be a global weak solution of (1.2) satisfying (1.9). As shown by the previous analysis, in terms of the variables t, β the quantities x, u, v satisfy the semil-linear system

$$\begin{cases} \frac{d}{dt}\beta(t, \beta_0) &= G(t, \beta(t, \beta_0)), \\ \frac{d}{dt}x(t, \beta(t, \beta_0)) &= u(t, \beta(t, \beta_0)), \\ \frac{d}{dt}u(t, \beta(t, \beta_0)) &= -P_x + kQ, \\ \frac{d}{dt}v(t, \beta(t, \beta_0)) &= 2(u^2 - P + kQ_x) \cos^2 \frac{v}{2} - \sin^2 \frac{v}{2}. \end{cases} \quad (6.28)$$

We recall that P, Q and G were defined at (1.3) and (6.8), respectively. The function P, P_x, Q and Q_x admit the representations in terms of β ,

$$\begin{aligned} P(x(\beta)) &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|\int_{\beta}^{\beta'} \cos^2 \frac{v(s)}{2} ds|} [u^2 \cos^2 \frac{v(\beta')}{2} + \frac{1}{2} \sin^2 \frac{v(\beta')}{2}] d\beta', \\ P_x(x(\beta)) &= \frac{1}{2} (\int_{\beta}^{+\infty} - \int_{-\infty}^{\beta}) e^{-|\int_{\beta}^{\beta'} \cos^2 \frac{v(s)}{2} ds|} [u^2 \cos^2 \frac{v(\beta')}{2} + \frac{1}{2} \sin^2 \frac{v(\beta')}{2}] d\beta', \\ Q(x(\beta)) &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|\int_{\beta}^{\beta'} \cos^2 \frac{v(s)}{2} ds|} u \cos^2 \frac{v(\beta')}{2} d\beta', \\ Q_x(x(\beta)) &= \frac{1}{2} (\int_{\beta}^{+\infty} - \int_{-\infty}^{\beta}) e^{-|\int_{\beta}^{\beta'} \cos^2 \frac{v(s)}{2} ds|} u \cos^2 \frac{v(\beta')}{2} d\beta'. \end{aligned}$$

For every $\beta_0 \in \mathbb{R}$, the initial data is in the following

$$\begin{cases} \beta(0, \beta_0) &= \beta_0, \\ x(0, \beta_0) &= x(0, \beta_0), \\ u(0, \beta_0) &= u_0(x(0, \beta_0)), \\ v(0, \beta_0) &= 2 \arctan(u_0)_x(x(0, \beta_0)). \end{cases} \quad (6.29)$$

By the Lipschitz continuity of all coefficients, the Cauchy problem (6.28)-(6.29) has a unique solution, globally defined for all $t \geq 0, x \in \mathbb{R}$. Finally, to complete the proof of uniqueness, we consider two solutions u, \tilde{u} of Eq.(1.2) with the same initial data $u_0 \in H^1(\mathbb{R})$. For a.e. $t \geq 0$ the corresponding Lipschitz continuous maps $\beta \mapsto x(t, \beta), \beta \mapsto \tilde{x}(t, \beta)$ are strictly increasing. Then, they have continuous inverses, saying $x \mapsto \beta^*(t, x), x \mapsto \tilde{\beta}^*(t, x)$. By the previous analysis, the map $(t, \beta) \mapsto (x, u, v)(t, \beta)$ is uniquely determined by the initial data u_0 . Therefore $x(t, \beta) = \tilde{x}(t, \beta)$ and $u(t, \beta) = \tilde{u}(t, \beta)$. In turn, for a.e. $t \geq 0$ this implies

$$u(t, x) = u(t, \beta^*(t, x)) = \tilde{u}(t, \tilde{\beta}^*(t, x)) = \tilde{u}(t, x).$$

□

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